Unboundedness of birational automorphisms

With E. Shinder

Plan

I Boundedness questions

II Motivic invariants and vanishing

III Non-vanishing and unboundedness

I Boundedness questions

Boundedness $\quad \mathbb{k}$: field

$\mathrm{Bir}_{\mathbb{k}} :=$ groupoid $\begin{cases} \text{Objects} : \mathrm{Var}/\mathbb{k} \\ \text{Morphisms} : \text{birational maps} \end{cases}$

$G \leq \mathrm{Bir}_{\mathbb{k}}$: (small) subgroupoid (e.g. $G = \mathrm{Bir}(X)$)

$\mathrm{Mor}_G \xrightarrow{\rho} \Sigma$ map to a set

Def A subset $H \subseteq \mathrm{Mor}_G$ (e.g. generating subsets)
is ρ-bounded if $\rho(H)$ is finite

Boundedness questions

$G \leq \mathrm{Bir}_{\mathbb{k}}$, $\quad \mathrm{Mor}_G \xrightarrow{\rho} \Sigma$

- Mor_G ρ-bounded ?
- $\exists$? ρ-bounded generating subsets

Examples

⓪ $G \leq \mathrm{Bir}_{\mathbb{k}}$ with $\mathrm{Mor}_G(X, Y) = \begin{cases} \emptyset & \text{if } X \neq Y \\ \mathrm{Id}_X & \text{otherwise} \end{cases}$

$\rightsquigarrow$ Boundedness questions of var. of certain types (BAB conjecture by Birkar, ...)

e.g. $G = \{\text{sm proj. complex CY } d\text{-fold}\} \xrightarrow{\chi} \mathbb{Z}$

bounded ?

<u>Examples</u> X : smooth proj. var / $\mathbb{k}$

① $\rho : \mathrm{Aut}(X) \to \mathrm{Aut}(X)/\mathrm{Aut}^\circ(X)$

$\exists$ ρ-bounded generators $\iff$ finitely generated

<u>Thm</u> (Lesieutre; Dinh, Oguiso, Yu...)

$\exists X$ (e.g. some Bl_{point} K3) s.t.

$\mathrm{Aut}(X)/\mathrm{Aut}^\circ(X)$ is not finitely generated

Examples X : smooth proj. var / $\mathbb{k}$

(2) $H \in \mathrm{Pic}(X)$ ample $\qquad d = \dim X$

$$\begin{aligned} \mathrm{Bir}(X) &\to \mathbb{Z}_{>0} \\ f &\mapsto \delta_{H,i}(f) := H^{d-i} \cdot f^* H^i \end{aligned}$$

often unbounded for $\langle f \rangle$

We consider the ith dynamical degree :

$$d_i(f) := \lim_{n \to \infty} \delta_i(f^n)^{\frac{1}{n}} \qquad \text{(Dinh-Sibony)}$$

which is moreover independent of H

Examples X : smooth proj. var / $\mathbb{k}$

② **Thm** (Diller - Favre) When $\dim X = 2$, $\forall f \in \mathrm{Bir}(X)$

$d_1(f)$ is an algebraic integer

$$\begin{aligned} \{ f \in \mathrm{Bir}(X) \mid X \in \Sigma \} &\overset{\rho}{\longrightarrow} \mathbb{Z}_{>0} \\ f &\longmapsto \deg d_1(f) \end{aligned}$$

$\Sigma \subset \{ \text{surfaces} \}$

Thm ρ is

- (Blanc - Cantat) bounded $(\underset{\text{char}=0}{\leq 20}, \underset{\text{char}>0}{22})$
  if $\Sigma = \{$ geom. irrat. surfaces $\}$
- (McMullen) unbounded if $\Sigma = \{$ rat surfaces $/ \mathbb{C} \}$

Groupoid of Mori fiber spaces

X : Fano variety / $\mathbb{k}$ with $\rho(X) = 1$ ($\mathbb{Q}$-fact. terminal)

$MF(X)$: groupoid defined by

- Objects : Mori fiber spaces $\dashleftarrow\dashrightarrow X$
- Morphisms : compositions of Sarkisov links

Sarkisov program (Corti, Hacon-McKernan)

$$\iff \quad \begin{aligned} MF(X) &\to \mathrm{Bir}_{\mathbb{k}} \\ [Y/S] &\mapsto Y \end{aligned} \quad \text{is full}$$

Examples

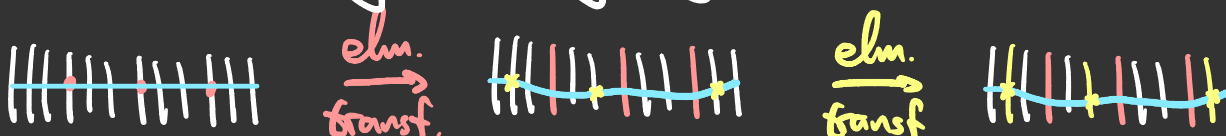
③ $\forall$ bir. map $Y \overset{f}{\dashrightarrow} Z$ between normal proj. surfaces

$n(f) := \#$ geometric irreducible components of $\mathrm{Exc}(f)$

Suppose $[\bar{\mathbb{k}} : \mathbb{k}] = \infty$, then $\exists\, f \in \mathrm{Bir}(\mathbb{P}^1 \times \mathbb{P}^1 / \mathbb{P}^1)$

having an irreducible exceptional divior

with arbitrarily many geometric irred. comp.



Consequence: The groupoid of rational surfaces / $\mathbb{k}$

is not generated by n-bounded map

Examples

③ $n(f) :=$ # geometric irreducible components of $\mathrm{Exc}(f)$

For surfaces :

$\{$ Sarkisov links not of Type II / curve $\} \xrightarrow{n} \mathbb{Z}_{\geq 0}$

is bounded ($\Leftarrow$ boundedness of weak dP surfaces)

n(Type II / curve) bounded $\Leftrightarrow$ $[\bar{\mathbb{k}} : \mathbb{k}] < \infty$

(Iskovskikh 96; Bernasconi - Fanelli - Schneider - Zimmermann '25)

Classification of Sarkisov links in dim 2

Examples (Frumkin, Lamy ; Blanc - Cheltsov - Duncan - Prokhorov)

④ $f: X \overset{\text{bir}}{\dashrightarrow} Y$ with X, Y sm. proj. $\mathbb{C}$ 3-folds

$\forall$ exceptional divisor E of f

$\exists!$ sm proj. curve C s.t. $E \dashrightarrow \mathbb{P}^1 \times C$

Def $g(f) = \max\{ g(C) \mid E \in \mathrm{ExcDiv}(f) \quad E \dashrightarrow \mathbb{P}^1 \times C \}$

$\mathrm{gon}(f) = \max\{ \mathrm{gon}(C) \mid \_\_\_\_\_\_\_\_\_\_\_\_ \}$

Thm (BCDP) X 3-fold $\dashrightarrow$ conic bundle. Then

both $g(\mathrm{Bir}(X))$ and $\mathrm{gon}(\mathrm{Bir}(X))$ are unbounded

Ex Spreading out the unbd construction in dim 2

Examples (Blanc - Cheltsov - Duncan - Prokhorov)

4. Def $g(f) = \max\{ g(C) \mid G \in \mathrm{ExcDiv}(f) \quad G \dashrightarrow \mathbb{P}^1 \times C \}$

$\mathrm{gon}(f) = \max\{ \mathrm{gon}(C) \mid \text{\_\_\_\_\_\_\_\_\_\_} \}$

Based on boundedness of weak Fano + 3-fold MMP:

(BCDP) $\{$ Sarkisov links $X/_Z \dashrightarrow Y/_T \mid \dim Z$ or $\dim T = 0 \}$

have bounded genus and gonality

Cor X: solid Fano 3-fold

Both $g(\mathrm{Bir}(X))$ and $\mathrm{gon}(\mathrm{Bir}(X))$ are bounded

Recapitulation / Goal of the talk

(I) In general, $\mathrm{Bir}(X)$ is not generated by maps with "bounded centers" : already when $\dim = 2$ with $[\bar{\mathbb{k}} : \mathbb{k}] = \infty$, and $\dim \geq 3$ over $\mathbb{C}$

(II) If we allow "cancellation of centers", we have boundedness when $\dim \leq 3$

(III) When $\dim \geq 4$, we have unboundedness, even after cancellation of centers

II Motivic invariants & vanishing

Motivic invariants (Lin-Shinder) $\mathbb{k}$: field

$\forall \quad f : X \overset{\text{bir}}{\dashrightarrow} Y$ between irreducible varieties / $\mathbb{k}$

$\mathrm{Ex}(f) := \{ x \in X^{(1)} \mid f \text{ is not an isomorphism at } x \}$

codim 1 points

Def

$$c(f) = \sum_{y \in \mathrm{Ex}(f^{-1})} [\bar{y}] - \sum_{x \in \mathrm{Ex}(f)} [\bar{x}] \in \mathbb{Z}[\mathrm{Bir}/\mathbb{k}]$$

set of irred. var/$\mathbb{k}$ modulo birational equiv.

Lem

$$c(X \overset{f}{\dashrightarrow} Y \overset{g}{\dashrightarrow} Z) = c(f) + c(g)$$

c and factorization centers X, Y : smooth projective

$$c_{SB} : \{\text{Bir. maps}/\Bbbk\} \xrightarrow{c} \mathbb{Z}[\mathrm{Bir}/_{\Bbbk}] \to \mathbb{Z}[SB/_{\Bbbk}]$$

Here $X \sim_{SB} Y \overset{\text{DEF}}{\iff} X \times \mathbb{P}^m \dashrightarrow Y \times \mathbb{P}^n$

Given a weak factorization of $f : X \overset{\text{bir}}{\dashrightarrow} Y$

i.e. $X \overset{\varphi_0}{\longleftrightarrow} X_1 \overset{\varphi_1}{\longleftrightarrow} \cdots \longleftrightarrow X_\ell \overset{\varphi_\ell}{\longleftrightarrow} Y$ (WF)

φ_i : blowup or blowdown along smooth centers

Additivity $\Rightarrow$ $c_{SB}(f) = \sum [T] - \sum [S] \in \mathbb{Z}[SB/_{\Bbbk}]$

T: blow-up centers S: blow-down centers

and it is independent of (WF)

Vanishing results

Despite the unboundedness phenomena we've seen earlier in dim 2, 3

Vanishing Thm $\mathrm{Bir}(X) \xrightarrow{c} \mathbb{Z}[\mathrm{Bir}/\Bbbk]$ is zero when

- (Lin-Shinder-Zimmermann '19) $\Bbbk$ perfect, $\dim X = 2$
- (LS '22, '25) $\Bbbk = \bar{\Bbbk}$ $\dim X = 3$

Prop (LS) Assume $\operatorname{char} \Bbbk = 0$ and X sm proj. RC 3-fold

$$c(\mathrm{Bir}(X)) \subset \mathbb{Z} \cdot \left\{ [C \times \mathbb{P}^1] - [C' \times \mathbb{P}^1] \;\middle|\; \begin{array}{l} C, C' \text{ sm. proj. irred. curves} \\ \text{s.t. } \mathrm{Jac}(C) \underset{\text{ppav}}{\simeq} \mathrm{Jac}(C') \end{array} \right\}$$

Torelli (Serre) $\Rightarrow$ only curves with $g(\text{geom. irred. comp}) \leq 1$ occur in $c(\mathrm{Bir}(X))$

Prop (LS) Assume char $\Bbbk = 0$ and X sm proj. RC 3-fold

$$c(\mathrm{Bir}(X)) \subset \mathbb{Z}\cdot\left\{[C\times\mathbb{P}^1]-[C'\times\mathbb{P}^1] \;\middle|\; \begin{array}{l} C, C' \text{ sm. proj. irred. curves} \\ \text{s.t. } \mathrm{Jac}(C) \underset{\text{ppav}}{\simeq} \mathrm{Jac}(C') \end{array}\right\}$$

Proof X, Y : sm proj. RC 3-folds

$$\mathrm{Bir}(X,Y) \xrightarrow{c} \mathbb{Z}\{C\times\mathbb{P}^1 \mid C \text{ curves}\} \xrightarrow{\mathrm{Jac}} K_0(\mathrm{PPAV}/\Bbbk)$$

$$f \longmapsto [J^3(Y)] - [J^3(X)]$$

blow-up formula for intermediate Jacobian (Murre, Benoist-Wittenberg)

$\{\mathrm{PPAV}/\Bbbk\}$ is a semi-simple abelian category $\Rightarrow$ $\square$

III Non-vanishing & unboundedness

Non-vanishing Thm (LS '22, '25) Assume char $\Bbbk = 0$

B geom. integral $\Bbbk$-var with $\dim B > 0$ (e.g. $\mathbb{P}^{n \geq 1}$)

$$c_{SB}(\mathrm{Bir}(\mathbb{P}^{N \geq 3}_{\Bbbk(B)})) \neq 0$$

Unboundedness Thm (LS '25) Same $\Bbbk$ and B

$X := \mathbb{P}^3 \times B$, $n := \dim X$.

$$\mathrm{Im}\left(\mathrm{Bir}(X) \xrightarrow{c} \mathbb{Z}[\mathrm{Bir}/\Bbbk] \xrightarrow{MRC} \mathbb{Z}[\mathrm{Bir}/\Bbbk]\right)$$

$[D] \longmapsto [\text{MRC-base of } D]$

contains a geometrically unbounded subgroup

$H \leq \mathbb{Z}[\mathrm{Bir}_{n-2}]$ bir. classes of $\dim = n-2$

Comments on the unboundedness Thm

Unboundedness Thm (LS '25) $X := \mathbb{P}^3 \times B$, $n := \dim X$. Then

$$\operatorname{Im}\left(\operatorname{Bir}(X) \xrightarrow{c} \mathbb{Z}[\operatorname{Bir}/\mathbb{k}] \xrightarrow{\text{MRC}} \mathbb{Z}[\operatorname{Bir}/\mathbb{k}]\right)$$
$$[D] \longmapsto [\text{MRC-base of } D]$$

contains a geometrically unbounded subgroup

$$H \leq \mathbb{Z}[\operatorname{Bir}_{n-2}]$$

Def $H \leq \mathbb{Z}[\operatorname{Bir}/\mathbb{k}]$ is called geometrically bounded if $\exists$ proper flat morphism $\mathcal{D} \to T$ of $\bar{\mathbb{k}}$-schemes of finite type s.t. $H_{\bar{\mathbb{k}}} \leq \langle \mathcal{D}_t \mid t \in T(\bar{\mathbb{k}}) \rangle \leq \mathbb{Z}[\operatorname{Bir}/\bar{\mathbb{k}}]$

Comments on the unboundedness Thm

Unboundedness Thm (LS '25) $X := \mathbb{P}^3 \times B$, $n := \dim X$. Then

$$\mathrm{Im}\left(\mathrm{Bir}(X) \xrightarrow{c} \mathbb{Z}[\mathrm{Bir}/k] \xrightarrow{\mathrm{MRC}} \mathbb{Z}[\mathrm{Bir}/k]\right)$$

$$[D] \mapsto [\text{MRC-base of } D]$$

contains a geometrically unbounded subgroup

$$H \leq \mathbb{Z}[\mathrm{Bir}_{n-2}]$$

$\forall f \in \mathrm{Bir}(X)$ with X smooth proper of dim n, exceptional divisors of f are ruled, so

$$\mathrm{MRC}(c(\mathrm{Bir}(X))) \subset \mathbb{Z}[\mathrm{Bir}_{\leq n-2}]$$

Non-vanishing for $c_{SB}(\mathrm{Bir}(\mathbb{P}^3))$ $\quad K$: field

Prop-NV C : smooth proj. curve of $g=1$ s.t. $C(K)=\emptyset$, $\mathrm{Jac}^5(C)(K)\neq\emptyset$ and $j(C)\neq 1728$. Then

$$[C]-[\mathrm{Jac}^2 C]\neq 0 \in c_{SB}(\mathrm{Bir}(\mathbb{P}^3_K))$$

Proof

$$\begin{tikzcd}
 & \mathrm{Bl}_C\mathbb{P}^4 \arrow[d] \arrow[r, phantom, "\simeq"] & \mathrm{Bl}_{\mathrm{Sym}^2 C}\mathbb{P}^4 \arrow[d] & \\
C \arrow[r, hook] & \mathbb{P}^4 \arrow[r, dashed, "|2H-C|"] & \mathbb{P}^4 & \mathrm{Sym}^2 C \arrow[l, hook'] \\
 & Q^3 \arrow[u, phantom, "\cup"] \arrow[r, dashed, "f"'] & \mathbb{P}^3 \arrow[u, phantom, "\cup"] &
\end{tikzcd}$$

C (Crauder-Katz)

$$\begin{tikzcd}
 & \mathrm{Bl}_C Q^3 \arrow[d] \arrow[r, phantom, "\simeq"] & \mathrm{Bl}_{\mathrm{Jac}^2(C)}\mathbb{P}^3 \arrow[d] \\
\mathbb{P}^3 \arrow[r, dashed, "\text{std. rat.}", "\text{constr.}"'] & Q^3 \arrow[r, dashed, "f"'] & \mathbb{P}^3
\end{tikzcd}$$

s

$$c_{SB}(f\circ s)=[C]-[\mathrm{Jac}^2 C]\neq 0\in \mathbb{Z}[SB/K] \qquad \square$$

Non-vanishing for $c_{SB}(\mathrm{Bir}(\mathbb{P}^3))$ K : field

Prop-NV C : smooth proj. curve of $g=1$ s.t. $C(K)=\emptyset$, $\mathrm{Jac}^5(C)(K) \neq \emptyset$ and $j(C) \neq 1728$. Then

$$[C] - [\mathrm{Jac}^2 C] \neq 0 \in c_{SB}(\mathrm{Bir}(\mathbb{P}^3_K))$$

Proof of $c_{SB}(\mathrm{Bir}(\mathbb{P}^{N\geq 3}_{k(B)})) \neq 0$ (Assuming $k = \bar{k}$) $K := k(B)$

$\exists$ elliptic curve E/K s.t. $j(E) \in k$ and $\mathbb{Z}/5 \leq E(K)$ (Cox-Parry)

Exact :

$$0 \to \underbrace{\frac{E(K)}{5\cdot E(K)}}_{\text{finite}} \to H^1(K, E[5]) \to H^1(K, E)[5] \to 0$$

(Lang-Néron) finite

$$H^1(K, E[5]) \xleftarrow{\text{f.g. ker}} H^1(K, \mathbb{Z}/5) \cong \mathrm{Hom}(\mathrm{Gal}_K, \mathbb{Z}/5) \ni \alpha \neq 0 \text{ infinite}$$

$\alpha \mapsto C \in H^1(K, E)[5]$

C is as in Prop-NV $\square$

Spreading out motivic invariants B : irred. var$/\mathbb{k}$

Suppose $\exists f \in \mathrm{Bir}(\mathbb{P}^3_{\mathbb{k}(B)})$ s.t.

$$c(f) = [C \times \mathbb{P}^1] - [C' \times \mathbb{P}^1] \in \mathbb{Z}[\mathrm{Bir}/\mathbb{k}(B)]$$

for some curves $C, C' / \mathbb{k}(B) \xrightsquigarrow{\text{spread}} \Sigma/_B, \Sigma'/_B$

$$f = \tilde{f} : \begin{tikzcd} \mathbb{P}^3 \times B \arrow[rr, dashed] \arrow[dr] & & \mathbb{P}^3 \times B \arrow[dl] \\ & B & \end{tikzcd}$$

$\exists\, U \underset{\neq \emptyset}{\subseteq} B$ s.t. all exc. div of $\begin{tikzcd} \mathbb{P}^3 \times U \arrow[rr, dashed, "\tilde{f}"] \arrow[dr] & & \mathbb{P}^3 \times U \arrow[dl] \\ & U & \end{tikzcd}$

dominate U

$$\rightsquigarrow \quad c(\tilde{f}) = [\Sigma \times \mathbb{P}^1] - [\Sigma' \times \mathbb{P}^1] \in \mathbb{Z}[\mathrm{Bir}/\mathbb{k}]$$

Unboundedness constructions

B geom. integral $\Bbbk$-var with $\dim B > 0$, $\operatorname{char} \Bbbk = 0$

The unbd thm is reduced to

Prop $\exists$ a sequence $\{\xi_d : J_d \to B\}_{d \geq 1}$ of ell. fib s.t.

1) ξ_d : non isotriv. with a section & $MW(\xi_d) \geq \mathbb{Z}/5$

2) ξ_d is an Iitaka fibration

3) $\operatorname{Bir}(J_{d,\bar{\Bbbk}}) = \operatorname{Bir}(J_{d,\bar{\Bbbk}}/B_{\bar{\Bbbk}})$

4) The finite part of the Stein fact. of the j-map $B \dashrightarrow \mathbb{P}^1$ of ξ_d has deg $\geq d$

Recapitulation

(I) In general, $\mathrm{Bir}(X)$ is not generated by maps with "bounded centers": already when $\dim = 2$ with $[\bar{\mathbb{k}} : \mathbb{k}] = \infty$, and $\dim \geq 3$ over $\mathbb{C}$

(II) If we allow "cancellation of centers", we have boundedness when $\dim \leq 3$

(III) When $\dim \geq 4$, we have unboundedness, even after cancellation of centers

Thank you !